for curve B. A comparison of the data of Figs. 3 and 4 reveals the improvement in average power handling capacity obtained by using a high Curie temperature tip of ferrite positioned on the load end. The trend of decreasing isolation loss at the low end of the band as the effective power is raised is markedly reduced in Fig. 4, where, for the composite ferrite isolator, isolation loss at 2.9 kmc is greater than 15 db for an effective average input power of 3200 watts. Without the high Curie temperature ferrite the isolator of Fig. 3 displays a reverse loss of <13 db at 2.9 kmc for an effective average input of only 1500 watts.

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Reflection Coefficient of E-Plane Tapered Waveguides*

In a paper by Matsumaru,[1] formulas of the input reflection coefficients of the linearly and sinusoidally E-plane tapered waveguides are given. Excellent agreements between the theoretical and experimental results have been found in both cases. In this note we wish to add some analytical remarks.

The analysis given in the above paper is different from the rigorous one given by Walker and Wax.[2] The latter led to a nonlinear differential equation

$$\frac{dR}{dx} - 2\gamma R + \frac{1-R^2}{2}\frac{d}{dx}\ln\left[Z(x)\right] = 0 \quad (1)$$

where R is the reflection coefficient, $Z(x)$ is the surge impedance of the tapered line, and $\gamma = \alpha + j\beta$ is the wave propagation constant. If the tapered line is loss-free, then we have $\gamma = j\beta$. On the assumption that the phase constant, β is independent of x, and that $R^2 \ll 1$, Bolinder[3] obtained an approximate expression of the input reflection coefficient

$$R = \frac{1}{2}\int_0^l \frac{d}{dx}\left[\ln Z(x)\right]\cdot e^{-j2\beta x}dx \quad (2)$$

for a finite tapered line of length l, terminated by $Z(0) = Z_1$ and $Z(l) = Z_2$ at each end.

It may be shown that Mr. Matsumaru's equations (4) and (12) are equivalent to (2) in this communication. On substitution of the surge impedance of a sinusoidal taper

$$Z(x) = \frac{Z_1 + Z_2}{2} - \frac{Z_1 - Z_2}{2}\cos\left(\frac{\pi x}{l}\right) \quad (3)$$

into our (2), we obtain his (12). Substituting the surge impedance of a linear taper

$$Z(x) = Z_1 + (Z_2 - Z_1)x/l \quad (4)$$

into our (2), and letting $x = y + l/2$, we obtain his (4), with its independent variable x being replaced by y. Therefore, it appears that Bolinder's assumption of $R^2 \ll 1$ should also apply to Mr. Matsumaru's analytical results. This is not, however, stated explicitly in his paper.

For a linearly tapered line defined by (4), (2) may be integrated exactly in terms of Ci and Si, the cosine and sine integrals. The input reflection coefficient is

$$R = \tfrac{1}{2}e^{ju_2}\{[Ci(u_2) - Ci(u_1)] - j[Si(u_2) - Si(u_1)]\} \quad (5)$$

where $u_1 = 2\beta l/(k-1)$, $u_2 = 2\beta lk/(k-1)$ and $k = Z_2/Z_1$. This expression appears to be somewhat simpler than Mr. Matsumaru's (8), and his (7), a binomial-expansion approximation, is not necessary in this case. If a change of variable, $u = 2\beta(q^{-1} + x)$, is made, his (5) leads directly to the above result—our (5).

In the treatment of a sinusoidally tapered line, noting that $r = (Z_1 - Z_2)/(Z_2 + Z_1)$ tends to zero first, and letting l tend to zero next, Mr. Matsumara showed how his (15) becomes

$$R = (Z_2 - Z_1)/(Z_2 + Z_1),$$

the reflection coefficient of two directly connected waveguides. It is felt that this statement, although correct, might mislead one to think that Matsumaru's (12) is exact. To clarify this point, we let l in his (15) tend to zero first and retain the higher order terms; (15) then becomes

$$\lim_{l\to 0} R = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right) + \frac{1}{3}\left(\frac{Z_2 - Z_1}{Z_2 + Z_1}\right)^3 + \cdots = \frac{1}{2}\ln\frac{Z_2}{Z_1}.$$

It is seen that as l tends to zero, R tends to $\frac{1}{2}\ln(Z_2/Z_1)$ rather than to $(Z_2 - Z_1)/(Z_2 + Z_1)$. This limiting case indicates somewhat the approximate nature of Matsumaru's (12), from which his (15) is derived. It might be said that the approximation becomes increasingly good as r tends to zero; then

$$\lim_{l\to 0} R = \frac{1}{2}\ln\frac{Z_2}{Z_1} \cong \frac{Z_2 - Z_1}{Z_2 + Z_1}.$$

It is also noted that our (5) also becomes $\frac{1}{2}\ln(Z_2/Z_1)$ as l tends to zero. As long as we use our (2) or its equivalent—Mr. Matsumaru's equations (4) and (12)—this is true, regardless of the nature of $Z(x)$ or type of taper. This can be seen directly from our (2), in which the phase factor tends to unity as l tends to zero. Direct integration gives the proof.

Eq. (2) in this demonstration may be considered as the first approximation of the solution to our differential equation (1), which—together with higher order approximations—has been discussed elsewhere.[4] In general it may be said that if the length of taper is longer than half of a guide-wavelength, the second order approximation has no significant effect.

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[1] K. Matsumaru, "Reflection coefficient of E-plane tapered waveguides," IRE Trans. on Microwave Theory and Techniques, vol. MTT-6, pp. 143–149, April, 1958.

[2] L. R. Walker and N. Wax, "Non-uniform transmission lines and reflection coefficients," *J. Appl. Phys.*, vol. 17, pp. 1043–1045; December, 1946.

[3] F. Bolinder, "Fourier transforms in the theory of inhomogeneous transmission lines," Proc. IRE, vol. 38, p. 1354; November, 1950.

[4] L. Solymar, "On higher order approximations to the solution of nonuniform transmission lines," Proc. IRE, vol. 45, pp. 1547–1548; November, 1957.

Author's Comment[5]

I am grateful to Mr. Yang for his remarks regarding my paper; his detailed remarks strengthen some of the weak points in it.

First, his analysis of his (4) is known to me, and I have no further comments to make on it. Next, his formula (5) is probably quite useful in calculating the reflection coefficient of linear tapers. In the latter part of his communication, he has made some remarks on the limiting cases of R. Although I had previously considered these analytical studies, I did not discuss them fully since they seemed to be too detailed for my paper.

As I mentioned in my paper, the main purpose was to present practical design data rather than detailed analyses. I would like to take this opportunity to add some comments on the experimental data described in my paper. Figs. 1 and 2, plotted in the K-plane,

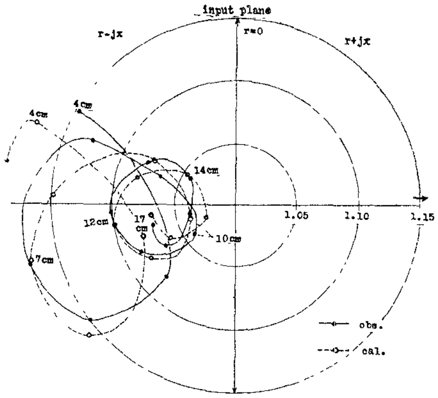

Fig. 1—Results of experiments, part I ($Z_2/Z_1 = 2.0$). Data are shown for linear-taper lengths from 4 to 17 cm.

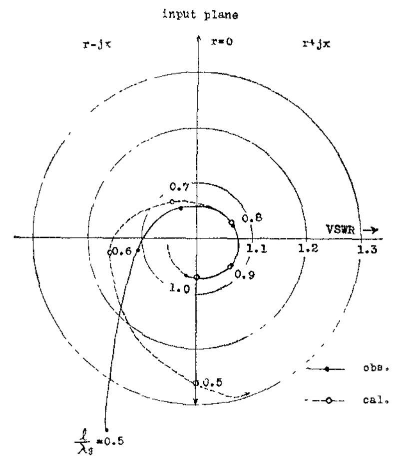

Fig. 2—Results of experiments, part III ($Z_1/Z_2 = 2.4$). The normalized sinusoidal-taper length l/λ_g was varied from 0.5 to 1.0.

show the reflection coefficients of the data obtained from experiments, parts I and III, respectively. The conically looped circular loci of R of the linear tapers in Fig. 1 show the typical behavior for the cases of $Z_2 > Z_1$. It should be mentioned that the position of R follows almost the course of one conical cycle every half-wavelength (4.9 cm). For

[5] Received by the PGMTT, August 4, 1958

the cases of $Z_1 > Z_2$, (experiments, part II), the loci of R are like open eccentric spirals about the center. The loci of sinusoidal tapers in Fig. 2 appear similar to a reduced concentric spiral about the center; these are typical for the cases of $Z_1 > Z_2$. The position of R also sweeps almost one cycle every half-wavelength. The locations in the K-plane and the forms of these loci are almost identical for several surge impedance ratios; hence these two figures depict typical characteristics for the general cases. Moreover, regarding the limiting cases of $l=0$ for linear tapers, I have obtained reasonable data for the behavior of R.

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The Permeability Matrix for a Ferrite Medium Magnetized at an Arbitrary Direction and Its Eigenvalues*

In analysis of propagation through magnetized ferrites it is usually assumed that the applied magnetostatic field is along one of the axes of the microwave carrier. It may be of interest to analyze the more general case; one in which the applied magnetostatic field is at an angle arbitrary to the axes of the microwave carrier.

If the geometry of Fig. 1, where $\boldsymbol{H}_0$ stands for the applied magnetostatic field and the carrier axes are x, y and z, is assumed, then, using the equation for the motion of the magnetization[1]

$$\frac{\partial \boldsymbol{M}}{\partial t} = \gamma \boldsymbol{M} \times \boldsymbol{H} \tag{1}$$

the following relation between the vector $\boldsymbol{B}$ and $\boldsymbol{H}$ results:

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \mu+(\mu_0-\mu)\sin^2\theta\cos^2\phi & \frac{\mu_0-\mu}{2}\sin^2\theta\sin 2\phi+j\kappa\cos\theta & \frac{\mu_0-\mu}{2}\sin 2\theta\cos\phi-j\kappa\sin\theta\sin\phi \\ \frac{\mu_0-\mu}{2}\sin^2\theta\sin 2\phi-j\kappa\cos\theta & \mu+(\mu_0-\mu)\sin^2\theta\sin^2\phi & \frac{\mu_0-\mu}{2}\sin 2\theta\sin\phi+j\kappa\sin\theta\cos\phi \\ \frac{\mu_0-\mu}{2}\sin 2\theta\cos\phi+j\kappa\sin\theta\sin\phi & \frac{\mu_0-\mu}{2}\sin 2\theta\sin\phi-j\kappa\sin\theta\cos\phi & \mu_0-(\mu_0-\mu)\sin^2\theta \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} \tag{2}$$

or, in short notation,

$$B = MH \tag{3}$$

where the usual approximations of small signal theory[1] have been used. One may observe that this completely general permeability matrix still preserves Hermitian character as long as losses are neglected; and, of course, it reduces to simple well-known forms in cases when $\boldsymbol{H}_0$ is along any of the axes of the microwave carrier.

It may be sometimes desirable to express the relation between the vectors $\boldsymbol{B}$ and $\boldsymbol{H}$ in a canonical form. This can be accomplished by finding the principal axes of the medium —or, to put it differently—by finding a coordinate system in which there exists a relation of the form

$$(B_\alpha') = (\lambda_{\alpha\alpha})(H_\alpha') \tag{4}$$

where B_α' and H_α' are the components of the magnetic induction and magnetic intensity along the axes of the new coordinate system, and $(\lambda_{\alpha\alpha})$ is a diagonal matrix composed of the eigenvalues of the permeability matrix of (2). The procedure of finding the components of the matrix $(\lambda_{\alpha\alpha})$ is usually referred to as an eigenvalue problem.[2] In our case it amounts to finding an unitary matrix P such that

$$PMP^{-1} = (\lambda_{\alpha\alpha})$$
$$P^{-1} = \tilde{P}^*. \tag{5}$$

The transformation of the components of the magnetic induction and the magnetic intensity from the original to the new coordinate system are

$$B' = PB$$
$$H' = PH. \tag{6}$$

To find the matrix P we solve the eigenvalue equation

$$(M - I\lambda)U = 0 \tag{7}$$

where U is a matrix composed of three row vectors from which the matrix P can be constructed by means of an orthonormalization process.[3] Eq. (7) has a unique solution only if the determinant

$$|M - I\lambda| = 0, \tag{8}$$

which yields the results

$$\lambda_{1,2} = \mu \pm \kappa$$
$$\lambda_3 = \mu_0. \tag{9}$$

The eigenvalues of (9) are exactly the same as they would be if the applied magnetostatic field were along any one of the coordinate axes of Fig. 1. This fact may be somewhat surprising.

The amount of algebra involved in finding the matrix P corresponding to the permeability matrix of (2) is prohibitive. We shall try a simpler but still general enough case in which the applied magnetostatic field is in the $x-y$ plane, *i.e.*, $\theta=\pi/2$ in Fig. 1. In such a case the permeability matrix becomes

$$M = \begin{bmatrix} \mu+(\mu_0+\mu)\cos^2\phi & \frac{\mu_0-\mu}{2}\sin 2\phi & -j\kappa\sin\phi \\ \frac{\mu_0-\mu}{2}\sin 2\phi & (\mu+(\mu_0-\mu)\sin^2\phi & j\kappa\cos\phi \\ j\kappa\sin\phi & -j\kappa\cos\phi & \mu \end{bmatrix}. \tag{10}$$

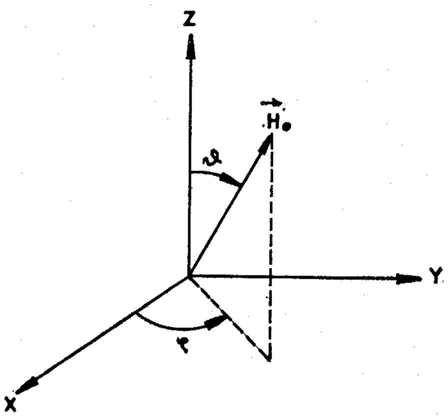

Fig. 1.

The corresponding unitary matrix P can be found to be

* Received by the PGMTT, September 12, 1958.

[1] D. Polder, "On the theory of electromagnetic resonance," *Phil. Mag.*, vol. 40, pp. 99–115; 1949.

[2] See, *e.g.*, H. Goldstein, "Classical Mechanics," Addison-Weseley Publishing Co., Inc., p. 119, 1956.

[3] H. Goldstein, *ibid.*, p. 328.